A recipe theorem for the topological Tutte polynomial of Bollobás and Riordan

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Abstract

In [BR01], [BR02], Bollobás and Riordan generalized the classical Tutte polynomial to graphs cellularly embedded in surfaces, i.e. ribbon graphs, thus encoding topological information not captured by the classical Tutte polynomial. We provide a 'recipe theorem' for their new topological Tutte polynomial, $R(G)$. We then relate $R(G)$ to the generalized transition polynomial $Q(G)$ of [E-MS02] via a medial graph construction, thus extending the relation between the classical Tutte polynomial and the Martin, or circuit partition, polynomial to ribbon graphs. We use this relation to prove a duality property for $R(G)$ that holds for both oriented and unoriented ribbon graphs. We conclude by placing the results of Chumutov and Pak [CP07] for virtual links in the context of the relation between $R(G)$ and $Q(R)$.

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1 Introduction

One of Thomas Brylawski's major contributions to the study of the Tutte polynomial was the development of what has come to be known as the 'recipe theorem'. It shows that any Tutte-Gröthendieck invariant must be an evaluation of the Tutte polynomial, with the necessary substitutions given by the recipe. This idea first appears in Brylawski's thesis [Bry70], with applications throughout much of his early work [Bry71, Bry72a, Bry72b]. Overviews and applications of the recipe theorem can be found in his work [Bry82], the comprehensive compilation by Brylawski and Oxley [BO92], and also in Oxley and Welsh [OW79], Welsh [Wel93], and the survey by Ellis-Monaghan and Merino [E-MMa, E-MMb].

The recipe theorem is essentially a universality statement, and as such, is a very valuable theoretical tool. Because of this, analogous results are sought for various generalizations of the Tutte polynomial (as well as other graph and matroid polynomials). Here we find a recipe theorem for a generalization of the Tutte polynomial given by Bollobás and Riordan.

In [BR01], [BR02], Bollobás and Riordan extended the classical Tutte polynomial to topological graphs, that is, graphs embedded in surfaces. In [BR01], Bollobás and Riordan defined the *cyclic graph polynomial*, a three variable contraction-deletion polynomial for graphs embedded in oriented surfaces. They furthered this work, using a different approach, in [BR02], with the *ribbon graph polynomial*, a four variable polynomial for graphs embedded in arbitrary surfaces that subsumes the three variable version. The ribbon graph polynomial, $R(G; x, y, z, w)$, is also sometimes called the *Bollobás-Riordan polynomial* or *topological Tutte polynomial*.

We provide a 'recipe theorem' that, analogously to that for the classical Tutte polynomial, establishes conditions for when a graph invariant can be calculated from the topological Tutte polynomial and gives a formula for this translation. It also restates of the universality property of $R(G; x, y, z, w)$ given by Bollobás and Riordan [BR02].

We show that if certain relations among the variables are satisfied, then the topological Tutte polynomial is related via an embedded medial graph to the generalized transition polynomial of [E-MS02]. This result extends the relation between the classical Tutte polynomial and the Martin polynomial given by Martin [Mar77] (cf. Jaeger [Jae90] and Las Vergnas [Las78, Las79, Las83]). It also extends the relation between the Tutte polynomial and the original transition polynomial given by Jaeger [Jae90].

We then use these results to give a duality property of $R(G; x, y, z, w)$ and applications to knots and links.

2 Preliminaries

We assume the reader is familiar with the work of Bollobás and Riordan in [BR01], [BR02] and we adopt the terminology therein, with the conventions of [BR02] taking precedence. We also assume the reader is familiar with cellular embeddings of graphs and with ribbon graphs (also known as fat graphs or band decompositions), and we generally follow Gross and Tucker [GT87]. Thus, we only briefly review a few essential concepts.

A cellular embedding of a graph in a surface (orientable or unorientable) can be specified by providing a sign for each edge and a rotation scheme for the set of half edges at each vertex, where a rotation scheme is simply a cyclic ordering of the half edges about a vertex. This is equivalent to a *ribbon graph*, which is a surface with boundary where the vertices are represented by a set of disks and the edges by ribbons, giving a half-twist to the ribbon of an edge with a negative sign. A ribbon graph can also be thought of as a *fat graph*, that is, a slight 'fattening' of the edges of the graph as it is embedded in the surface, or equivalently a 'cutting out' of the graph, together with a small neighborhood of it, from the surface.

Fig. 1 shows a graph with two vertices and two parallel edges, one positive and one negative. It is embedded on a Klein bottle, and the ribbon graph is a Möbius band with boundary.

For a ribbon graph G, we let $k(G), r(G)$, and $n(G)$ be, respectively, the number of connected components, rank, and nullity of the underlying abstract graph. Additionally, $bc\,(G)$ is the number of boundary components of the surface defining the ribbon graph G, and $t(G)$ is an index of the orientability of the surface, with $t(G) = 0$ if the surface is orientable, and $t(G) = 1$ if it is not. When $A \in E(G)$, then $r(A)$, $\kappa(A)$, $n(A)$, $bc(A)$ and $t(A)$, each refer

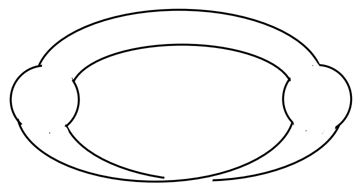

Figure 1: A ribbon graph which is a Möbius band with boundary, which can be viewed as a neighborhood of the graph as embedded in a Klein bottle.

to the spanning subgraph of G on the edge set A, with embedding inherited from G.

The result of deleting an edge from a ribbon graph is clear. For contraction of a non-loop edge e, assume the sign of e is positive, by flipping one endpoint if necessary to remove the half twist (this reverses the cyclic order of the half edges at that vertex and toggles their signs). Then G/e is formed by deleting e and identifying its endpoints into a single vertex v. The cyclic order of half edges at v follows first the original cyclic order at one endpoint, beginning where e had been, and continuing with the cyclic order at the other endpoint, again beginning where e had been. The surface that results from sewing disks to the boundaries of a ribbon graph G may not be the same surface that result from sewing disks to the boundary of $G - e$ or G/e, particularly when e is a bridge or a loop, i.e. $G - e$ or G/e are not necessarily embedded in the same surface as G.

There are two definitions of the topological Tutte polynomial, a generating function formulation and a linear recursion formulation, that were shown to be equivalent by Bollobás and Riordan [BR02]. We begin with the generating function formulation.

Definition 2.1. *Let G be a ribbon graph. Then*

$$R(G;x,y,z,w) = \sum_{A\subseteq E(G)} (x-1)^{r(G)-r(A)} y^{n(A)} z^{\kappa(A)-bc(A)+n(A)} w^{t(A)}$$
$$\in Z[x,y,z,w]/\langle w^2 - w\rangle.$$

The linear recursion formulation derives from the following theorem.

Theorem 2.2. *[BR02] If G is a ribbon graph, then*

$$R(G;x,y,z,w) = R(G/e;x,y,z,w) + R(G-e;x,y,z,w),$$

if e is an ordinary edge; and

$$R(G;x,y,z,w) = x\,R(G/e;x,y,z,w),$$

if e is a bridge.

Repeated application of this theorem reduces a ribbon graph to a disjoint union of embedded *bouquets*, that is, embedded graphs each consisting of a single vertex with some number of loops, and the topological information is distilled into these minors of the original graph.

Signed chord diagrams are a useful device for determining the relevant parameters of an embedded bouquet. A *signed chord diagram* is a circle with n symbols each appearing twice on its perimeter, with a signed (± 1) chord drawn between each pair of like symbols. We assign a symbol to each loop of an embedded bouquet G and arrange them on the perimeter of the circle in the chord diagram in the same order as the cyclic order of the half-edges about the vertex, with a chord receiving the same sign as the loop it represents. Since signed chord diagrams are exactly equivalent to bouquets, we will use the terms interchangeably. If we 'fatten' the chords as in Fig. 2, with a negative chord receiving a half-twist, then $bc(G) = bc(D)$, the number of components in the resulting diagram. Similarly, since G has only one vertex, $n(G)$ is the number of edges of G, which is the number of chords of D, so we denote this by $n(D)$. We also set $t(D) = t(G)$, and note that $t(D) = 0$ if all chords of D have a positive sign, and $t(D) = 1$ otherwise. This, combined with Definition 2.1, gives in Proposition 2.3 the necessary evaluations of the terminal forms to complete the linear recursion formulation.

Proposition 2.3. *If G is an embedded bouquet with corresponding signed chord diagram D, then*

$$R(G;x,y,z,w) = \sum_{D' \subseteq D} y^{n(D')} z^{1-bc(D')+n(D')} w^{t(D')},$$

where the sum is over all subdiagrams D' of D.

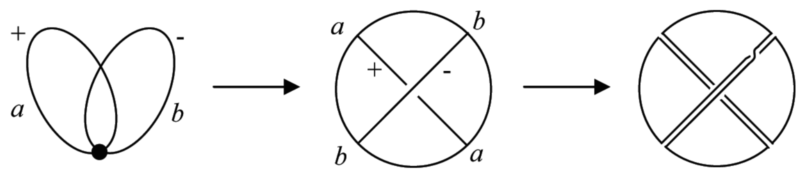

Figure 2: An embedded bouquet with two loops, one positive and one negative, with its signed chord diagram and the boundary components of the fattened signed chord diagram.

3 The recipe theorem

In this section we give the recipe theorem that specifies precisely when and how a function may be recovered from $R(G;x,y,z,w)$. It is essentially a restatement of the universality of $R(G;x,y,z,w)$ from Bollobás and Riordan [BR02] in a form that facilitates its application.

Following [BR02] we say that two chord diagrams are related by a *rotation* about the chord e if they are related as D_1 and D_2 in Figure 3. Two chord diagrams are related by a *twist* about e if they are related as D_3, D_4 in Fig. 3, where a letter represents a sequence of labels about the circle, and a prime symbol means to reverse the order of the sequence. Two diagrams are *related* if they are related by a sequence of rotations and twists. From [BR02] we have that any diagram is related to a *canonical signed chord diagram* D_{ijk} ($0 \leq k \leq 2$) that consists of $i-2j-k$ positive chords intersecting no other chord, j pairs of intersecting positive chords and k negative chords intersecting no other chord.

We also observe that

$$R(D_{100};x,y,z,w) = 1+y, \tag{1}$$

$$R(D_{101};x,y,z,w) = 1+yzw, \tag{2}$$

$$\text{and } R(D_{210};x,y,z,w) = y^2z^2+2y+1. \tag{3}$$

Since from [BR02], $R(G;x,y,z,w)$ is multiplicative on one-point joins of ribbon graphs, where the new rotation is given by simply concatenating the rotation systems of the vertices being joined, then

$$R(D_{ijk}) = [R(D_{100})]^{i-2j-k}[R(D_{210})]^j[R(D_{101})]^k. \tag{4}$$

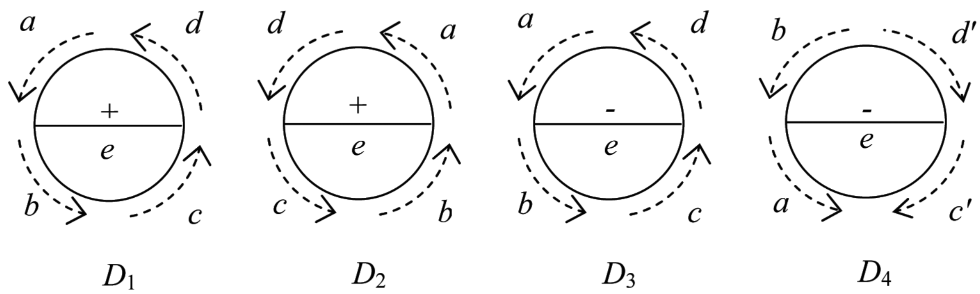

Figure 3: Related chord diagrams where the dotted arrows indicate directions of sequences of labels along the perimeter.

Furthermore, Bollobás and Riordan [BR02] show that if D is equivalent to D_{ijk} then $R(D) = R(D_{ijk})$.

Theorem 3.1. *(The recipe theorem.)*

Let F be a map from a minor closed subset $\mathcal{M}$ of ribbon graphs containing D_{210}, D_{100}, and D_{101} to a commutative ring $\mathcal{R}$ with unity; let $s = F(D_{210})$, $q = F(D_{100})$ and $r = F(D_{101})$; and suppose there are elements $\alpha, x, u, v \in R$ with α a unit such that:

1. $$F(G) = \begin{cases} F(G/e) + F(G \setminus e) \text{ if } e \text{ is ordinary }, \\ xF(G/e) \text{ if } e \text{ is a bridge}; \end{cases}$$

2. *$F(G\dot{\cup}H) = F(G)F(H)$ and $\alpha F(G * H) = F(G)F(H)$ where G, H are embedded bouquets, $G\dot{\cup}H$ is the disjoint union, and $G * H$ is the one-point join, again with concatenated rotation system;*

3. *$F(\mathbf{E}) = \alpha^n$ if $\mathbf{E}$ is an edgeless graph on n vertices;*

4. *$(q-\alpha)^2u^2 = \alpha(s - 2q + \alpha)$, and $(q-\alpha)uv = r - \alpha$, and also $v = v^2$.*

Then

$$F(G) = \alpha^{k(G)} R(G; x, \alpha^{-1}q - 1, u, v),$$

where $k(G)$ is the number of components of G.

Proof. The proof is by a double induction, first on the number of chords in a signed chord diagram, and then on the number of non-loop edges of G.

We first note that by Item 2 and 4 and Equations 1 through 4, the result holds for any canonical signed chord diagram.

Since this recipe theorem is also a universality statement, unsurprisingly the proof uses the same central observations about chord diagrams as the proof of universality for Theorem 2 from Bollobás and Riordan [BR02]. Thus, from [BR02], we have that since F satisfies Item 1, it satisfies

$$F(D_1) - \mu F(D_1^{'}) = F(D_2) - \mu F(D_2^{'}) \text{ and} \tag{5}$$

$$F(D_3) - \mu F(D_3^{'}) = F(D_4) - \mu F(D_4^{'}), \tag{6}$$

where the D_i's are related as in Figure 3 with $D_i^{'} = D_i - e$, and $\mu = 1$ if there is a chord from $a \cup b$ to $c \cup d$, and otherwise $\mu = x$.

The same identities hold for $R(G; x, y, z, w)$, and hence hold for $F' = F - \alpha R$.

Now suppose $G = D$ is a signed chord diagram corresponding to an embedded bouquet. Assume by induction that $F(D) = \alpha R(D; x, \alpha^{-1}q - 1, u, v)$ if D is a signed chord diagram with fewer than m chords. Then $F'(G)$ vanishes on signed chord diagrams with fewer than m chords.

This, with Equations 5 and 6, implies that $F'(D) = F'(D')$ if D and D' are related chord diagrams with m chords. In particular, D is related to a canonical diagram D_{ijk}, so $F'(D) = F'(D_{ijk})$.

Thus

$$F'(D) = F'(D_{ijk}) = F(D_{ijk}) - \alpha R(D_{ijk}; x, \alpha^{-1}q - 1, u, v) = 0,$$

and hence by induction, $F'(D) = 0$ on all signed chord diagrams. This extends to disjoint unions of embedded bouquets by Item 2 and that $R(G; x, y, z, w)$ is also multiplicative on disjoint unions.

Thus the result holds for all ribbon graphs with no non-loop edges. If G has a non-loop edge e, the result is immediate by induction from Item 1, observing that, if e is a bridge, then $k(G/e) = k(G)$.

$\square$

In analogy to the classical case, we call a function on ribbon graphs satisfying the conditions of Theorem 3.1 a *topological Tutte invariant*, and

the theorem itself justifies calling the ribbon graph polynomial of Bollobás and Riordan *the* topological Tutte polynomial.

We give a quick example by applying Theorem 3.1 to give the relationship noted in [BR02] between $R(G; x, y, z, w)$ and the oriented graph invariant $C(G; x, y, z)$ of [BR01]. Cyclic graphs form a minor-closed subset of ribbon graphs, and we extend C very slightly by defining $C(D_{101}; x, y, z) = 1 + yz^{\frac{1}{2}}w$, noting that the domain is still minor-closed. If we let $\mathcal{R}$ be the quotient ring $\mathbb{Z}[x, y, z^{\frac{1}{2}}, w]/\langle w - w^2\rangle$, then C satisfies the recipe theorem with $\alpha = 1$, $1 + 2y + y^2 z = s$, $1 + y = q$, $1 + yz^{\frac{1}{2}}w = r$ and taking $u = z^{\frac{1}{2}}$ and $v = w$. Thus, $C(G; x, y, z) = R(G; x, y, z^{\frac{1}{2}}, w)$.

The polynomial R has the property that it immediately identifies whether or not a ribbon graph is oriented, just by the absence or presence of the one variable w. For an arbitrary topological Tutte invariant however, this property is sensitive to the structure of the ring in which it takes its values.

Corollary 3.2. *If F, $\mathcal{M}$, $\mathcal{R}$ satisfy Theorem 3.1, with both $q - \alpha$ and $r - \alpha$ being units of $\mathcal{R}$, then v=1, and thus F does not discern orientation by the presence or absence of a single idempotent element.*

Proof. From Item 4, we have that $\frac{r-\alpha}{q-\alpha} = uv = uv^2 = v\frac{r-\alpha}{q-\alpha}$. Thus, since $q - \alpha$ and $r - \alpha$ are units, $v = 1$, and hence $F(G) = \alpha^{k(G)} R(G; x, \alpha^{-1}q - 1, u, 1)$. □

This corollary raises a number of questions. Suppose F is a topological Tutte invariant with the properties of Corollary 3.2. Then F does not determine orientation by the presence or absence of one idempotent element. However, unless u also equals 1, F can still distinguish between oriented and unoriented embeddings of a graph. For example, it is easy to check that $R(G, x, y, z, 1)$ distinguishes all the canonical chord diagrams, and furthermore a chord diagram D is orientable if and only if a term of the form yz does not appear in $R(D; x, y, z, 1)$. Since this is so, is w strictly necessary to record orientability information? I.e. F can obviously be computed from R, but is it possible to recover R from some such F? We suspect not, since a consistent translation from F to R is problematic even on the canonical chord diagrams, which leads to the next question. Is R actually more refined than any such F? I.e. is there a pair of ribbon graphs distinguished by R that are not distinguished by F? Finally, the most basic question is whether it is always possible to determine from some such F that a ribbon graph G is oriented.

4 Medial graphs and transition polynomials

The classical Tutte polynomial, $T(G;x,y)$, among many other properties, encodes information about families of Eulerian circuits in the medial graph of a planar graph. This theory is the result of a relation between the classical Tutte polynomial and the Martin, or circuit partition, polynomial. Here we extend this theory to ribbon graphs, giving an analogous result relating the topological Tutte polynomial of a ribbon graph to the transition polynomial of its topological medial graph, where the transition polynomial of [E-MS02] is a multivariable generalization of the circuit partition polynomial. The original relation between the Tutte polynomial and the Martin polynomial can be found in Martin's 1977 thesis [Mar77], with the theory considerably extended by Martin [Mar78], Las Vernas [Las79, Las83, Las88], Jaeger [Jae90], Bollobás [Bol02], and [E-M00, E-M04a, E-M04b, E-MS02]. An overview can be found in Ellis-Monaghan and Merino [E-MMa, E-MMb].

The *medial graph* of a connected planar graph G is constructed by placing a vertex on each edge of G and drawing edges around the faces of G. The faces of this medial graph are colored black or white, depending on whether they contain or do not contain, respectively, a vertex of the original graph G. This face two-colors the medial graph. The edges of the medial graph are then directed so that a black face is to the left of each edge. This directed medial graph is denoted $\vec{G}_m$, and is an *Eulerian digraph*, that is, the number of incoming edges is equal to the number of outgoing edges at each vertex.

For the circuit partition polynomial we first recall that an *Eulerian vertex state* is a choice of reconfiguration at a vertex of an Eulerian digraph $\vec{G}$. The reconfiguration consists of replacing a $2n$-valent vertex v with n 2-valent vertices joining pairs of edges originally adjacent to v, where each incoming edge must be paired with an outgoing edge. An *Eulerian graph state* of an Eulerian digraph $\vec{G}$ is the result of choosing one vertex state at each vertex of $\vec{G}$. Note that a graph state is a disjoint union of consistently oriented cycles. Let $\mathcal{S}$ denote the set of Eulerian graph states of an Eulerian digraph $\vec{G}$, where $\mathcal{S}$ is not up to isomorphism, so that each individual state is included in the set.

The *circuit partition polynomial* of an Eulerian digraph is $j\left(\vec{G};x\right) = \sum\limits_{S\in\mathcal{S}} x^{c(S)}$, where $c(S)$ is the number of components of the state S. For a connected planar graph G with oriented medial graph $\vec{G}_m$, the relations among the Martin polynomial, circuit partition polynomial, and classical Tutte poly-

nomial are

$$x^{k(G)}m(\vec{G}_m; x+1) = j(\vec{G}; x) = x^{k(G)}T(G; x+1, x+1). \qquad (7)$$

In the context of transition polynomials such as the circuit partition polynomial (and also certain link invariants), the number of components of a state does not count isolated vertices. Thus, we use c here for components, in contrast with the k we use in the context of Tutte polynomials where isolated vertices are included in the component count.

To extend Equation (7) to ribbon graphs, we begin with the notion of a topological medial graph. Let G be a connected ribbon graph, thought of as being cellularly embedded in a surface. We construct the medial graph G_m in the same surface, exactly as in the planar case. That is, we place a vertex of G_m on each edge of G, and draw the edges of G_m by following two adjacent half-edges of G around the face they bound, as in Figure 4, which shows the topological medial graph G_m, where G is a single loop with a negative edge. Both G_m and G are embedded in a Klein bottle. Note that if e is a negative edge, two of the half edges which are consecutive with the e between them receive a half twist. Whether or not an edge of G_m is positive or negative depends on the parity of the half twists on its half edges.

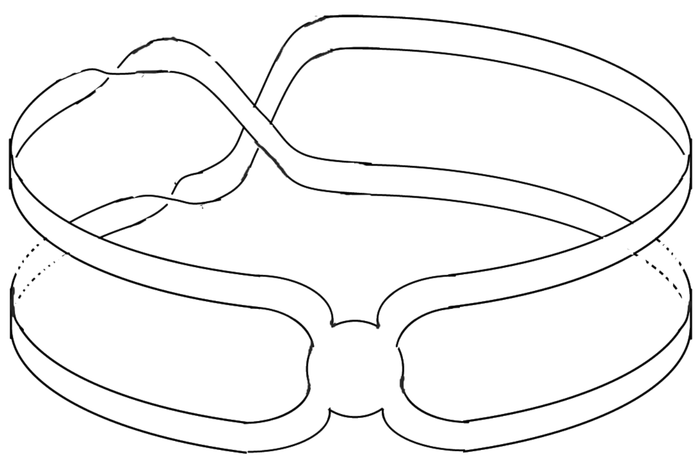

Figure 4: The topological medial graph of a single loop with a negative edge.

We define the medial graph of an isolated vertex to be a "free loop",

that is an edge, but no vertex, following the boundary of a small disk on the surface containing the vertex.

The generalized transition polynomial, $Q(G;W,t)$ of [E-MS02] is a multivariable extension of the circuit partition polynomial that assimilates the transition polynomial of Jaeger [Jae90] for 4-regular graphs and generalizes it to arbitrary Eulerian graphs. We recall the essentials needed for the current application, and refer the reader to [E-MS02] for full details.

A *weight system*, $W(G)$, of an Eulerian graph G is an assignment of a pair weight in a unitary ring $\mathcal{R}$ to every possible pair of adjacent half edges of G. (We simply write W for $W(G)$ when the graph is clear from context.) A *pair weight* is the particular value $p\left(e_v, e_v'\right)$ associated by the weight system to a pair of half edges e_v, e_v' incident with a vertex v. The *vertex state weight* of a vertex state is $\prod p(e_v, e_v')$ where the product is over the pairs of half edges comprising the vertex state. The *state weight* of a graph state S of a graph G with weight system W is $\omega(S) = \prod \omega(v,S)$, where $\omega(v,S)$ is the vertex state weight of the vertex state at v in the graph state S, and where the product is over all vertices of G.

The generalized transition polynomial is defined exactly like the circuit partition polynomial, simply with the addition of keeping track of the specific weights for each pair of adjacent edges given by the weight system.

Definition 4.1. *Let G be a graph having weight system W with values in a unitary ring $\mathcal{R}$. Then the generalized transition polynomial is $Q(G;W,t) = \sum_{S\in\mathcal{S}} \omega(S)t^{c(S)}$, where $c(S)$ is the number of components of the state S.*

In the case that G is a planar graph with oriented medial graph $\vec{G_m}$, we can assign a weight system W to the underlying medial graph G_m, with pair weights of 1 for each pair of half edges where one is incoming and the other outgoing in $\vec{G_m}$, and 0 otherwise. With this weight system, $Q(G_m;W,x) = j(\vec{G_m};x)$.

For the current application, we will restrict Q to medial ribbon graphs. In the rotation system about a vertex v of a medial graph G_m of a graph G, we can consider *six* half edges, the four half edges actually belonging to G_m, plus the two half edges of the edge of G corresponding to v. This allows us to define the following weight system, which we will refer to hereafter as the *medial weight system.* If a pair of edges of G_m are consecutive in the rotation system at v without a half edge of G between them, we assign them a pair weight of $\sqrt{\alpha}$. If they are consecutive with a half edge of G between

them, we assign them a pair weight of $\sqrt{\beta}$. Otherwise, their pair weight is zero. The square root is just a notational convenience so that the vertex state weights of the two possible nonzero vertex states will be either α or β. This refinement to the level of pair weights is not strictly necessary to the current paper, but we provide it since it is required for splitting formulas of Q in [E-MS02] that may be useful in future applications. We can think of these two vertex states as either leaving a ribbon of G intact, or 'snipping' through it, and so we will refer to them as 'uncut' or 'cut' vertex states, respectively. See Figure 5.

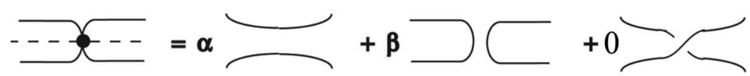

Figure 5: The weight system for a topological medial graph. On the left is a neighborhood of a vertex of the topological medial graph G_m with the corresponding edge of G as a dotted line. On the right are the three possible vertex states, uncut, cut, and crossing, with their respective vertex state weights.

We now give the relationship between the generalized transition polynomial and the topological Tutte polynomial of Bollobás and Riordan that extends the classical case. Although the proof technique here is very similar to that of Moffatt [Mof08] and Chmutov and Pak [CP07], the result is much broader, since the link invariants they address are specializations of the generalized transition polynomial. See Section 6.

Theorem 4.2. *Let G be a ribbon graph with topological medial graph G_m, and let G_m have the medial weight system W. Then*

$$Q(G_m; W, t) = \alpha^{r(G)} \beta^{n(G)} t^{k(G)} R\left(G; \frac{\beta t}{\alpha} + 1, \frac{\alpha t}{\beta}, \frac{1}{t}, 1\right).$$

Proof. Observe that if W is the medial weight system, then $Q(G_m; W, t) = \sum_{S \in \mathcal{S}} \alpha^{a(S)} \beta^{b(S)} t^{c(S)}$, where $a(S)$ and $b(S)$ are the number uncut or cut vertex states, respectively, in the graph state S. Thus, we can use the edges of G

to index this sum, and thinking of H as the set of edges that are uncut, we have that

$$Q(G_m;W,t) = \sum_{H\subset E(G)} \beta^{|E|-|H|}\alpha^{|H|}t^{bc(H)} = \beta^{|E|}\sum_{H\subset E(G)} \left(\frac{\alpha}{\beta}\right)^{|H|} t^{bc(H)}.$$

We then note that

$$R(G;,x,y,z,1) = \\ (x-1)^{-k(G)}y^{-v(G)}z^{-v(G)}\sum_{H\subset E(G)} \left((x-1)yz^2)\right)^{k(H)}(yz)^{|H|}z^{-bc(H)}.$$

Substituting $z=\frac{1}{t}$, $y=\frac{\alpha t}{\beta}$ and $x=\frac{\beta t}{\alpha}+1$ yields the result.

□

Alternatively, Theorem 4.2 could also have been proved as easily using Theorem 3.1.

5 Duality

The results of Section 4 provide tools to determine properties of $R(G;x,y,z,w)$.

We write G^* for the dual ribbon graph of a ribbon graph G (see Gross and Tucker [GT87] or Bollobás and Riordan [BR02]), and note that G^*_m and G_m are isomorphic ribbon graphs. Furthermore, G_m is a four regular ribbon graph that determines the same surface as G and G^* do. The cellular embedding requirement implies that all of G, G^*, G_m, andG^*_m have the same number of components.

If G_m is a topological medial graph with weight system W, then the *dual weight system* W^* results from exchanging the roles of α and β. This leads to the following duality relation for the generalized transition polynomial, the central idea for which is illustrated in Figure 6.

Theorem 5.1. *If G is a ribbon graph with dual G^*, then*

$$Q(G^*_m;W(G^*_m),t) = Q(G_m;W^*(G_m),t).$$

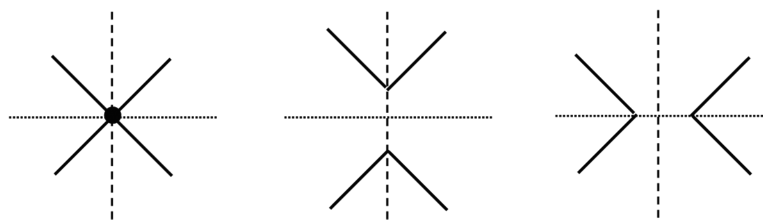

Figure 6: The lefthand figure shows a vertex and four half edges of G_m, or equivalently G_m^*, (solid) with the edge of G (dotted) and G^* (dashed). The center figure shows the vertex state which is uncut with respect to G and cut with respect to G^*. The righthand figure shows the vertex state which is cut with respect to G and uncut with respect to G^*.

Proof. If we think of $H \subset E(G)$ as indexing the uncut edges, then

$$Q(G_m; W^*(G_m), t) = \sum_{H \subset E(G)} \alpha^{|E|-|H|}\beta^{|H|}t^{bc(H)}.$$

However, as in Figure 6, an uncut vertex state of G_m corresponds to a cut vertex state of G_m^* and vice versa. Also $bc(G|_H) = bc(G^*|_{E-H})$. Thus, $Q(G_m; W^*(G_m), t) = \sum_{H \subset E(G^*)} \alpha^{|E|-|H|}\beta^{|H|}t^{bc(E-H)}$, where we think of $H \subset E(G^*)$ now as indexing the cut edges of $G*$. But if we instead index over sets of *uncut* edges, this is then $\sum_{H \subset E(G^*)} \beta^{|E|-|H|}\alpha^{|H|}t^{bc(H)} = Q(G_m^*; W(G_m^*), t)$. □

We can now give a duality relation that extends the duality relation for $R(G; x, y, z, w)$ given by Bollobás and Riordan in [BR02] from one degree of freedom to two, thus giving a natural extension of the duality of the classical Tutte polynomial. This theorem was first announced in [E-MS05] and has since been referenced by Moffatt [Mof, Mof08], Chmutov [Chm], and Vignes-Tourneret [V-T]. It is stronger than the version in [Mof08] in that it applies to unoriented as well as oriented ribbon graphs. Chmutov [Chm] gives an alternative proof and slightly different formulation.

Theorem 5.2. *Let G be cellularly embedded in a not necessarily connected surface Σ, let G^* be its dual, and let $\gamma = 2\sum g_i + \sum g_i$, where the first sum*

is of the genera of the orientable components of Σ *and the second sum is of the genera of the unorientable components. Then*

$$\beta^{\gamma}R(G^*;\frac{\beta t}{\alpha}+1,\frac{\alpha t}{\beta},\frac{1}{t},1)=\alpha^{\gamma}R(G;\frac{\alpha t}{\beta}+1,\frac{\beta t}{\alpha},\frac{1}{t},1). \tag{8}$$

Furthermore, if we write $\sqrt{t^2}$ *as* t*, and* $\sqrt{\frac{\beta^2}{\alpha^2}}$ *as* $\frac{\beta}{\alpha}$*, then we may substitute* $x=\frac{\beta t}{\alpha}$ *and* $y=\frac{\alpha t}{\beta}$ *to rewrite this as*

$$x^{\frac{\gamma}{2}}R(G;1+x,y,\frac{1}{\sqrt{xy}},1)=y^{\frac{\gamma}{2}}R(G^*;1+y,x,\frac{1}{\sqrt{xy}},1).$$

If $\alpha=\beta=1$*, then Equation (8) reduces to the one variable duality identity given in [BR02].*

Proof. Theorems 4.2 and 5.1 give that

$$\alpha^{r(G^*)}\beta^{n(G^*)}t^{k(G^*)}R(G^*;\frac{\beta t}{\alpha}+1,\frac{\alpha t}{\beta},\frac{1}{t},1)=\beta^{r(G)}\alpha^{n(G)}t^{k(G)}R(G;\frac{\alpha t}{\beta}+1,\frac{\beta t}{\alpha},\frac{1}{t},1).$$

To simplify the exponenets, we use the invariance of the Euler characteristic on each connected component of Σ, namely that $v_i-e_i+f_i=2-\gamma_i$, where γ_i is $2g_i$ or g_i depending orientability, and where v_i, e_i, and f_i are the number of vertices, edges, and faces, respectively, of G on the i^{th} component. By definition and duality, $r(G^*)=v(G^*)-k(G^*)=f(G)-k(G)$. Then $f(G)-k(G)=\sum_i f_i+\sum_i 1$, where the sum is over the number of components of Σ. By the invariance of the Euler characteristic on each component, this becomes $\sum_i(2-\gamma_i+e_i+v_i-1)=k(G)+e(G)-v(G)-\gamma$. Thus, $r(G^*)+\gamma=n(G)$, and similarly, $r(G)+\gamma=n(G^*)$, which gives the result. ∎

6 Applications to links

We now turn to the virtual links of Kauffman [Kau99] and Goussarov, Polyak, and Viro [GPV00]. Chmutov and Pak [CP07] found a relation between the Kauffman bracket of virtual links and $R(G)$. We will focus on their result for signed graphs, since it subsumes their unsigned version. Here we show that since the Kauffman bracket is another specialization of the generalized

transition polynomial, $Q(G)$, the results of [CP07] follow immediately from those of Section 4.

Let Σ be a compact oriented surface. Here we view a virtual link L as an unoriented link in $\Sigma \times [0,1]$, with link diagram $\tilde{L}$ in Σ, such that the link universe Γ_L is cellularly embedded in Σ (and hence is a ribbon graph). In a plane drawing of L, the crossings corresponding to actual crossings are called *classical*, and the others, artifacts of the projection, are called *virtual*. Following Kamada [Kam02, Kam04], a plane drawing of a virtual link is *checkerboard colorable* if a small neighborhood of one side of each strand can be colored so that a checkerboard pattern is formed at classical crossings, while the strand coloring passes through virtual crossings unchanged. See Figure 7. We also recall that an A splitting at a classical crossing is the result of opening a channel between the two regions swept out by rotating the top strand counterclockwise, and a B splitting joins the other two regions.

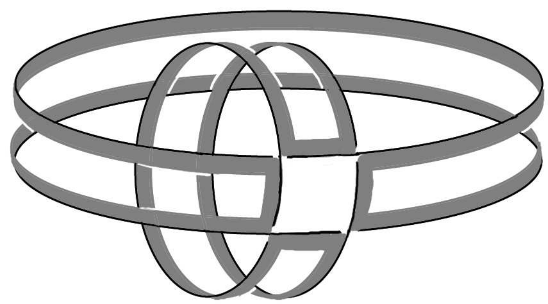

Figure 7: A checkerboard coloring of a plane drawing of a four component link in the torus. The four crossings forming a square in the center are *classical*, while the rest are *virtual*.

Definition 6.1. *The* generalized Kauffman bracket *of a virtual link diagram* $\tilde{L}$ *is the polynomial*

$$[\tilde{L}](A, B, d) = \sum_{S \in S(\tilde{L})} A^{a(S)} B^{b(S)} d^{c(S)-1},$$

where $S(\tilde{L})$ *is the set of states of* $\tilde{L}$*, where* $a(S)$ *and* $b(S)$ *are the number of* A *and* B *splittings, respectively, in the state* S *and where* $c(S)$ *is the number*

of components.

If Γ_L is the universe of a virtual link digram $\tilde{L}$, then it inherits a weight system, W_L, from $\tilde{L}$ by assigning a pair weight of $\sqrt{A}$ to half edges which are joined in an A splitting and a pair weight of $\sqrt{B}$ to those joined in a B splitting. With this, the generalized Kauffman bracket of any link diagram $\tilde{L}$ is a specialization of the generalized transition polynomial. The following theorem is a natural extension of Jaeger's [Jae90] relation between the original transition polynomial and Kauffman bracket.

Theorem 6.2. *Let L be a link in $\Sigma \times [0,1]$ with link diagram $\tilde{L}$ and universe Γ_L. Then $[\tilde{L}](A,B,d) = \frac{1}{d}Q(\Gamma_L; W_L, d)$*

Proof. This follows immediately from comparing Definitions 6.1 and 4.1. ☐

Chmutov and Pak [CP07] consider *signed ribbon graphs*, denoted $\hat{G}$, where the sign on the edges acts as a device to keep track of the over/under crossings of a not necessarily alternating link diagram. These are not the signed edges used to encode topological information in unoriented ribbon graphs defined previously. In this context, all ribbon graphs are oriented as topological surfaces. However, each edge has a +/- indicator associated with it. Chmutov and Pak [CP07] extend Definition 2.1 to these signed ribbon graphs as follows.

Definition 6.3. *Let $\hat{G}$ be a signed ribbon graph, and let F_1 be the number of negative edges in F and F_2 be the number of negative edges in $E(G)-F$. Then*

$$R(\hat{G};x,y,z) = \sum_{F\subseteq E(G)} (x-1)^{r(G)-r((F)+s((F)} y^{n((F)-s((F)} z^{k((F)-bc((F)+n((F)},$$

where $s(F) = \frac{1}{2}(F_1 - F_2)$.

Thus, if every edge of an signed oriented ribbon graph $\hat{G}$ is positive, then $R(\hat{G};x,y,z) = R(G;x,y,z,1)$ where G is the underlying unsigned ribbon graph, and $R(G;x,y,z,1)$ is as in Definition 2.1.

If $\hat{G}$ is a signed oriented ribbon graph with medial graph G_m, then we can define a signed weight system W^- by reversing the roles of α and β at vertices of G_m corresponding to negative edges of $\hat{G}$.

With this, we have the following signed analog to Theorem 4.2, with a virtually identical proof that we leave to the reader.

Theorem 6.4. *Let $\hat{G}$ be a signed oriented ribbon graph with medial graph G_m. Then*

$$Q(G_m; W^-, t) = A^{r(G)} B^{n(G)} t^{k(G)} R_{\hat{G}}(\frac{Bt}{A} + 1, \frac{At}{B}, \frac{1}{t}).$$

Proposition 6.5. *If $\tilde{L}$ is a checkerboard colorable link diagram in an oriented surface Σ, with universe Γ_L, then Γ_L is the medial graph G_{Lm} for some G_L. Furthermore, the link diagram induced weight system W_L of Γ_L is precisely the signed medial weight system W^- of G_{Lm} with α replaced by A and β replaced by B.*

Proof. The checkerboard coloring is exactly a face 2-coloring, say green and white, of the universe Γ_L in Σ, with the green faces bounded by the half edges that are joined by an A splitting. Thus, Γ_L is the medial graph of the green-face graph, which we denote G_L. We then note that the link diagram induced weight system W_L of Γ_L is precisely the medial weight system W of G_{Lm} with α replaced by A and β replaced by B, since the half edges paired by an A splitting(respectively B splitting) are precisely those with pair weight α (respectively β). ☐

The medial graph construction of Proposition 6.5 provides a natural interpretation of the gluing procedure used by Chmutov and Pak [CP07] to produce G_L.

An alternative proof for the main theorem of Chmutov and Pak [CP07] then follows immediately.

Theorem 6.6 (Chmutov and Pak, [CP07])**.** *If $\tilde{L}$ is checkerboard colored link diagram with signed green-face graph $\hat{G}_L$, then*

$$[\tilde{L}](A, B, d) = A^{r(\hat{G}_L)} B^{n(\hat{G}_L)} d^{k(\hat{G}_L)-1} R_{\hat{G}_L}(\frac{Bd}{A} + 1, \frac{Ad}{B}, \frac{1}{d}).$$

Proof. This is a consequence of Theorems 6.2 and 6.4 and Proposition 6.5. ☐

Dedication

This paper is dedicated to Tom Brylawski, who had a major influence on the first author, beginning in graduate school and continuing throughout her professional life. He has given us a profound and lasting legacy of deep mathematics.

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